

Chapter 7.4 part 1

Section 7.4 Isomorphisms and Homomorphisms

A map which respects the structure - homomorphism

Def Let G and H be groups.

A function (map) $f: G \rightarrow H$

such that $f(ab) = f(a)f(b)$ for every $a, b \in G$

is called a (group) homomorphism

} A morphism in the category of groups

Ex trivial $f: G \rightarrow G$ - identity map

$$g \mapsto g$$

$$f: G \rightarrow G$$

$$g \mapsto e$$

$$a, b \in \mathbb{Z}$$

$$\text{Ex } f: \mathbb{Z}_5^* \rightarrow \mathbb{Z}_5^*$$

$$x \mapsto x^2$$

$$f([a][b]) = f([ab]) = [(ab)^2] = [a^2 b^2]$$

$$= [a^2][b^2] = f([a])f([b])$$

$$f(x) = x^2 \quad \left\{ \begin{array}{l} f([x]) = [x^2] \end{array} \right.$$

$$[1] \mapsto [1]$$

$$[2] \mapsto [4]$$

$$[3] \mapsto [4]$$

$$[4] \mapsto [1]$$

$$[3^2] = [9] = [4] \quad 9 \equiv 4 \pmod{5}$$

$$[4^2] = [16] = [1] \quad 16 \equiv 1 \pmod{5}$$

Def Isomorphism is a bijective (injective & surjective) homomorphism

If there is an isomorphism $f: G \rightarrow H$, we say that G and H } $G \cong H$
are isomorphic

Prop The map

$f^{-1}: H \rightarrow G$ is a homomorphism

(therefore also an isomorphism)

Ex cyclic group of order 4

$$\begin{aligned} & \text{H} = \{e, a, a^2, a^3\} \cong \mathbb{Z}_4 \quad (\text{addition} \\ & \quad \text{modulo 4}) \\ & a \mapsto [1] \end{aligned}$$

$$\mathbb{Z}_5^* \cong \mathbb{Z}_4$$

$$[2] \mapsto [1]$$

$$[2][2] = [4] \mapsto [2] = [1] + [1]$$

$$[2][4] = [3] \mapsto [3] = [1] + [2]$$

$$[2][3] = [1] \mapsto [0] = [1] + [3]$$

$$\mathbb{Z}_5^* = \langle [2] \rangle = \langle [3] \rangle$$

$$\neq \langle [4] \rangle \subset \mathbb{Z}_5^*$$

proper subgroup

$$\langle [4] \rangle = \{[1], [4]\}$$

Th 7.19 Let G be a cyclic group

(1) if G is infinite, then $G \cong \mathbb{Z}$

(2) if G is finite, then $G \cong \mathbb{Z}_n$ with $n = |G|$

$$\mathbb{C} \times \mathbb{H} \log: \mathbb{R}^{**} \rightarrow \mathbb{R} \quad \mathbb{R}^{**} \cong \mathbb{R}$$

$$\log(ab) = \log(a) + \log(b)$$

Ex 9 G - a group; $c \in G$ - inner automorphisms

$$f: G \rightarrow G \quad f(g) = c^{-1}g c$$

$$g \mapsto c^{-1}g c \quad f(ab) = f(a)f(b)$$

$$f(ab) = \underline{c^{-1}abc}$$

$$f(a) = c^{-1}ac \quad f(b) = c^{-1}bc$$

$$f(a)f(b) = c^{-1}a\underline{c^{-1}bc}$$

$$= \underline{c^{-1}abc}$$

Terminology

an isomorphism from a group to itself
is called automorphism

General properties of homomorphisms

Thm 7.20 Let G, H be groups. $e_G \in G, e_H \in H$ - their identity elements

Let $f: G \rightarrow H$ be a (group) homomorphism. Then:

$$(1) f(e_G) = e_H$$

(3) $\text{Im } f$ is a subgroup in H

$$(2) f(a^{-1}) = f(a)^{-1}$$

$\text{Im } f = \{h \in H \mid \text{there is } g \in G \text{ s.t. } f(g) = h\}$

(4) If f is injective, then $G \cong \text{Im } f$

